How to observe the Efimov effect

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We propose to observe the Efimov effect experimentally by applying an external electric field on atomic three-body systems. We first derive the lowest order effective two-body interaction for two spin zero atoms in the field. Then we solve the three-body problem and search for the extreme spatially extended Efimov states. We use helium trimers as an illustrative numerical example and estimate the necessary field strength to be less than 2.7 V/Å.

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Introduction. The Efimov effect [1] is an interesting anomaly in a three-body system with short-range interactions. If at least two of the binary subsystems have an s-state at zero energy (or equivalently an infinitely large scattering length) the three-body system develops an infinite series of bound states. The binding energies of these states are exponentially small and the spatial extension is correspondingly exponentially large [2].

A peculiar feature of these states is that their number decreases when the modulus of the scattering length is decreased by either weakening or *strengthening* the two-body potentials. For a given scattering length a and effective range r_e the number of bound three-body states is proportional to $\log(|a|/r_e)$. This number becomes infinitely large also when the effective range approaches zero, which in this limit is called the Thomas effect [3].

Although the Thomas effect has only theoretical interest, the Efimov effect might be observed experimentally in a suitable weakly bound three-body system. Candidates for the Efimov effect are Borromean (that is without bound subsystems) halo nuclei, occurring at the neutron drip-line [4], and molecular systems like the atomic helium trimers [5–7]. For the ^{4}He-trimer two bound states are predicted with the excited state resembling an Efimov state. The ratio $|a|/r_e$ is in this case around 14. The excited state disappears when the interaction is either weakened by 3% or strengthened by 20%. So far other examples are not available since the condition of the exceedingly large scattering length is hard to meet.

This difficulty could be overcome if one could gradually alter the two-body potentials, thus adjusting the scattering length to the needed condition. An already large scattering length is very sensitive to small variations of the potential and the needed alteration should therefore be relatively small. Such tuning of the scattering length in an external magnetic field was recently suggested for a system of two rubidium atoms [8]. A corresponding Feshbach resonance at zero energy was subsequently observed experimentally [9]. Another similar resonance has also been observed for a system of two sodium atoms [10]. However corresponding investigations of three-body systems do not exist. We have therefore undertaken the first investigation of a three-body system in an external field with the main emphasis on the Efimov effect.

So far the most promising candidate for the Efimov effect is believed to be the system of three helium atoms. For the noble atoms, unlike the alkali atoms, the electric rather than magnetic field seems to be most suitable instrument. In this letter we suggest to use an external electric field as a tool to modify the interaction between noble atoms. We shall consider examples of helium isotopes and estimate the electric field strength needed to reach the Efimov limit. We shall investigate the problem of three helium atoms in an external electric field and estimate the properties of the lowest Efimov states. We shall also discuss the feasibility of an experiment observing the Efimov effect in atomic or molecular three-body systems.

Three-body system in an external field. An external field breaks the rotational symmetry and a precise computation of the Efimov states is exceedingly difficult. It is already difficult for systems with well defined angular momentum [4]. We shall therefore assume that our binary subsystems are already close to the threshold and that the needed external field is weak enough to justify a perturbative treatment of the problem.

We can not, however, apply the perturbation theory directly to the three-body problem since notwithstanding its weakness, this perturbation supposedly leads to a significant modification of the spectrum – appearance of an infinite series of bound states.

On the other hand on the two-body level this perturbation only slightly modifies the binary interactions. Our strategy is therefore to estimate perturbatively the variation of the potential between atoms exposed to an external field and then solve the three-body problem with the modified potentials.

Correction to the two-body potential. We shall now calculate the total energy of the system of two atoms exposed to a weak static external electric field. One of the terms in this energy is, within the usual adiabatic approximation, the sought correction to the potential.

The Hamiltonian of a system of two atoms separated by a fixed distance $\mathbf{r}$ in an external static electric field $\mathcal{E}$ can be written in the dipole approximation as

$$H = H_0^{(1)} + H_0^{(2)} + \Delta H$$
$$\Delta H = -\mathbf{d}^{(1)}\cdot\mathcal{E} - \mathbf{d}^{(2)}\cdot\mathcal{E} \quad (1)$$

$$+\frac{\mathbf{d}^{(1)}\cdot\mathbf{d}^{(2)}-3(\mathbf{d}^{(1)}\cdot\hat{\mathbf{r}})(\mathbf{d}^{(2)}\cdot\hat{\mathbf{r}})}{r^3} ,$$

where $H_0^{(i)}$ and $\mathbf{d}^{(i)}$ respectively are the unperturbed Hamiltonian and the dipole operator of the atom i ($i = 1, 2$). The unperturbed state of the atom i with the principal quantum number n_i, angular momentum l_i and its projection m_i on the direction of $\hat{\mathbf{r}} \equiv \mathbf{r}/|\mathbf{r}|$ is denoted by $|n_i l_i m_i\rangle$. The corresponding energy is E_{n_i} with the ground state energy set by definition to zero. As a shorthand notation we shall use $\nu_i = n_i l_i m_i$ and $|\nu_1\nu_2\rangle = |\nu_1\rangle|\nu_2\rangle$. The ground states will be denoted as $|0_i\rangle$ and $|0\rangle = |0_1\rangle|0_2\rangle$.

The operators $\mathbf{d}^{(i)}$ have negative parity and therefore the first order correction to the energy $\Delta E^{(1)} = \langle 0|\Delta H|0\rangle$ is zero since $\langle 0_i|\mathbf{d}^{(i)}|0_i\rangle = 0$. The second order correction

$$\Delta E^{(2)} = -\sum_{\nu_1\nu_2}\frac{\langle 0|\Delta H|\nu_1\nu_2\rangle\,\langle \nu_1\nu_2|\Delta H|0\rangle}{E_{n_1}+E_{n_2}} \qquad (2)$$

can be rewritten in terms of the reduced dipole matrix element d_{n_i} defined by

$$\langle n_i l_i m_i|\mathbf{d}^{(i)}\cdot\mathcal{E}|0_i\rangle \equiv \delta_{l_i,1} d_{n_i}\mathcal{E}_{m_i} , \qquad (3)$$

where $\mathcal{E}_m$ ($m = 0, \pm1$) is the usual spherical tensor component of the vector $\mathcal{E}$. By using this in Eq.(2) we get

$$\Delta E^{(2)} = -{\sum_{\nu_1}}'\frac{|d_{n_1}\mathcal{E}_{m_1}|^2}{E_{n_1}} - {\sum_{\nu_2}}'\frac{|d_{n_2}\mathcal{E}_{m_2}|^2}{E_{n_2}}$$
$$-\frac{1}{r^6}{\sum_{\nu_1\nu_2}}'\frac{|d_{n_1}d_{n_2}(1-3\delta_{m_1,0})|^2}{E_{n_1}+E_{n_2}}\delta_{m_1,-m_2} , \qquad (4)$$

where the primed summation sign indicates that only states with $l_i = 1$ are included. The first two terms are the usual polarization terms $-\frac{1}{2}\beta_i\mathcal{E}^2$, where the polarizability β_i in our notation is given by [11]

$$\beta_i = 2{\sum_{n_i}}'\frac{|d_{n_i}|^2}{E_{n_i}} . \qquad (5)$$

These terms can be effectively used to guide single atoms in electrostatic lenses [12]. However they do not change the interaction between atoms as they only give rise to a constant shift of the total energy. The last term in Eq.(4) is the long-range dipole-dipole part of the bare Van der Waals interaction without the external field.

The lowest order correction, caused by the field $\mathcal{E}$, to the two-body interaction appears in the third order perturbation term

$$\Delta E^{(3)} = \sum_{\nu_1\nu_2}\sum_{\nu_1'\nu_2'}(E_{n_1}+E_{n_2})^{-1}(E_{n_1'}+E_{n_2'})^{-1}$$
$$\times\langle 0|\Delta H|\nu_1\nu_2\rangle\langle\nu_1\nu_2|\Delta H|\nu_1'\nu_2'\rangle\langle\nu_1'\nu_2'|\Delta H|0\rangle . \qquad (6)$$

Using the definition in Eq.(3) we then get

$$\Delta E^{(3)} = \frac{4}{r^3}{\sum_{\nu_1\nu_2}}'\frac{|d_{n_1}|^2|d_{n_2}|^2}{E_{n_1}E_{n_2}}$$
$$\times\mathcal{E}_{m_1}\mathcal{E}_{m_2}\left(1-3\delta_{m_1,0}\right)\delta_{m_1,-m_2}$$
$$= \beta_1\beta_2|\mathcal{E}|^2\frac{1-3\cos^2\theta}{r^3} = -\beta_1\beta_2|\mathcal{E}|^2\sqrt{\frac{16\pi}{5}}\frac{Y_{20}(\theta)}{r^3} , \qquad (7)$$

where θ is the angle between $\mathbf{r}$ and $\mathcal{E}$. This term is clearly the interaction between two classical dipoles $\mathbf{D}^{(i)} = \beta_i\mathcal{E}$ induced by the field on the atoms.

Higher order corrections have higher powers in either field or polarizability and they are presumably smaller. However, the fourth order correction, with the third power of polarizability, includes a possibly important term which is proportional to $|\mathcal{E}|^2$, similarly to $\Delta E^{(3)}$. We shall estimate this small term approximately by use of the semiclassical operator

$$\delta H = \frac{\mathbf{D}^{(1)}\cdot\mathbf{d}^{(2)}-3(\mathbf{D}^{(1)}\cdot\hat{\mathbf{r}})(\mathbf{d}^{(2)}\cdot\hat{\mathbf{r}})}{r^3}$$
$$+\frac{\mathbf{d}^{(1)}\cdot\mathbf{D}^{(2)}-3(\mathbf{d}^{(1)}\cdot\hat{\mathbf{r}})(\mathbf{D}^{(2)}\cdot\hat{\mathbf{r}})}{r^3} . \qquad (8)$$

Each term in this operator corresponds to the interaction of an atom with a classical dipole $\mathbf{D}$.

The second order perturbation theory now gives for the first term in Eq.(8)

$$\delta E_1 = -{\sum_{\nu_2}}'\frac{|d_{n_2}(1-3\delta_{m_2,0})D^{(1)}_{m_2}|^2}{E_{n_2}r^6}$$
$$= -\frac{1}{2}\beta_2|\mathbf{d}^{(1)}|^2\frac{1+3\cos^2\theta}{r^6}$$
$$= -\beta_1^2\beta_2|\mathcal{E}|^2\sqrt{4\pi}\frac{Y_{00}(\theta)+\frac{1}{\sqrt{5}}Y_{20}(\theta)}{r^6} . \qquad (9)$$

The contribution δE_2 from the second term in Eq.(8) is directly obtained from Eq.(9) by interchanging the atomic indices 1 and 2.

In total we shall use the following correction to the two-body potential due to the external field

$$\Delta V(r) = \Delta E^{(3)} + \delta E_1 + \delta E_2$$
$$= -\beta_1\beta_2\sqrt{\frac{16\pi}{5}}\frac{Y_{20}(\theta)}{r^3}|\mathcal{E}|^2$$
$$-\beta_1\beta_2(\beta_1+\beta_2)\sqrt{4\pi}\frac{Y_{00}(\theta)+\frac{1}{\sqrt{5}}Y_{20}(\theta)}{r^6}|\mathcal{E}|^2 . \qquad (10)$$

Both terms in this two-body interaction are of second order in the external field. They differ in the radial dependence and the order of the polarizability. The extra polarizability factor and the faster radial fall off of the second term produce a relatively smaller contribution. The terms with Y_{20} break the rotational symmetry and couple different angular momenta. The individual properties of the two-body system determine the relative contribution of the two terms. Since both terms resulted from the second order perturbation treatment they must lower the ground state energy of the two-body system.

Numerical examples. To observe the Efimov effect introduced by an external field we have to select a three-body system where at least two of the binary subsystems are almost bound in relative s-states and the third subsystem is unbound. The atomic helium trimers seem to offer an interesting and realistic possibility where the necessary properties are present and accurately known, i.e. $^4He_2$ is weakly bound and $^3He^4He$ is marginally unbound [5–7]. Thus the $^4He^3He_2$ system is a realistic candidate where the external field then may provide the additional energy needed to bind $^3He^4He$. It would probably be even more advantageous to aim for three identical bosons.

We use the local and central potential LM2M2 as the free-space interaction between He atoms [13] and the polarizability is $\beta = 0.2050 \times 10^{-24}\text{cm}^3 = 1.383$ a.u. [11].

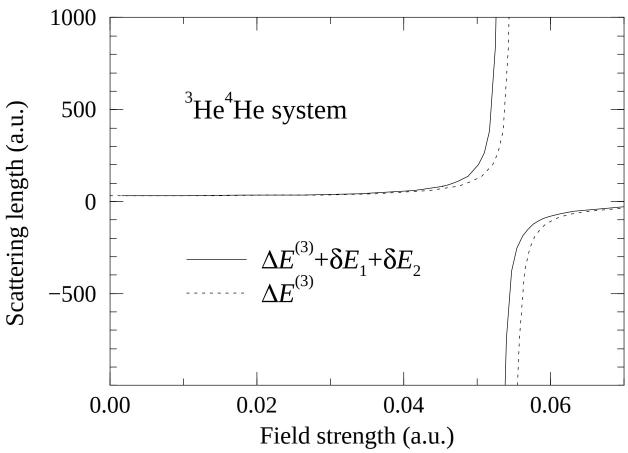

FIG. 1. The scattering length for the atomic $^3He^4He$ system as function of the strength of the external electric field. The free-space interaction is LM2M2 [13]. The induced interaction is from Eq.(10) where the solid curves include all terms and the dotted curves only include the first term.

Although the effective two-body interaction in Eq.(10) breaks the rotational symmetry, the decisive s-wave scattering length can still be calculated as function of the strength of the field. The result for $^3He^4He$ is shown in Fig. 1, where an infinite scattering length appears at $\mathcal{E} = 0.053$ a.u. Thus an s-state at zero energy is produced by this external field at this point, i.e. the Efimov conditions are fulfilled for the three-body system $^4He^3He_2$. In Fig.1 we also see that the last term in Eq.(10) is rather small for the present set of parameters. Other systems with larger polarizability may produce a zero energy s-state for substantially smaller fields.

For the $^3He_2$ system the scattering length is roughly constant (≈ 14 a.u.) over the range of the field strength in Fig. 1. The related divergence and the bound s-state appears at $\mathcal{E} = 0.067$ a.u. Such strong fields are possible to obtain with todays femto-second lasers [14].

Accurate computations are unfortunately rather difficult even without an external field [5–7]. However, to assess the possibility of the effect an order of magnitude estimate should be made before more elaborate efforts are mobilized. The three-body bound state energies can be obtained with a smaller numerical effort and a relative accuracy better than 50% by use of simple attractive gaussian two-body interactions with the correct s-wave scattering lengths [7]. We choose realistic range parameters and for a given strength of external field adjust the depth of the gaussian to reproduce the scattering lengths shown in Fig. 1 for $^3He^4He$. The accuracy of the resulting three-body binding energies relative to the strength of the potential is around 10^{-3}.

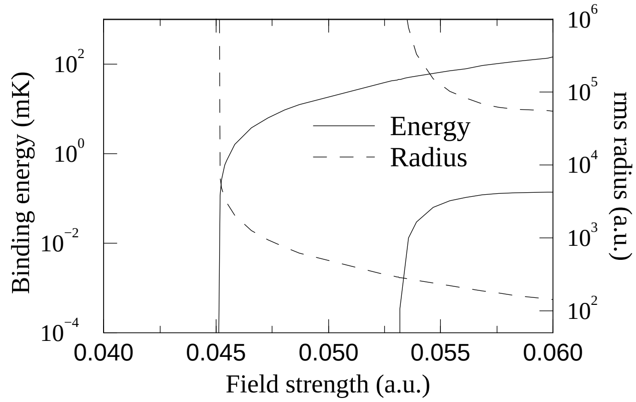

FIG. 2. The binding energies (solid curves) and root mean square radii (dashed curves) for the ground and first excited states of $^4He^3He_2$ as functions of the strength of the external field. The two-body interactions are $V_0(\mathcal{E}) \exp(-r^2/b^2)$, where $b = 10.546$ a.u. for $^4He^3He$ and 11.265 a.u. for $^3He^3He$. The strengths $V_0(\mathcal{E})$ produce the same s-wave scattering length as the LM2M2 potential with the field $\mathcal{E}$.

The resulting calculated sizes and binding energies of the ground and first excited states for $^3He^4He_2$ are shown in Fig. 2 as functions of the strength of the external field. Below $\mathcal{E}= 0.045$ a.u. the three-body system is unbound. Above this three-body threshold the ground state binding energy increases quickly to a level of about 0.1 K. Just below the two-body $^3He^4He$ threshold ($\mathcal{E}= 0.053$ a.u.) the first excited state appears with a binding energy quickly increasing to about 1 mK. The root mean square radii correspondingly decrease from infinity at the thresholds to about 10 a.u. and 200 a.u., respectively. Infinitely many bound three-body states with behavior similar to these lowest states must appear at the two-body threshold.

An electric field of 0.053 a.u.=2.7 V/Å induces a dipole moment of 0.074 a.u. which corresponds to the displacement of the center of the electron cloud by approximately 0.04 a.u. from the nucleus. This might involve a substantial change of the electronic structure of the atom and the original Van der Waals interaction would therefore also change. However it is not unlikely that a more accurate calculation, consistently including these effects, would produce Efimov conditions already for a weaker

external field.

Observing the effect. Better suited systems can probably be found. However almost inevitably two identical particles are needed, since their identical scattering lengths against the third particle can then be tuned simultaneously. The two-body subsystems should all be unbound but as near as possible to the threshold for binding. Two atoms of naturally occurring isotopes are almost always bound in a dimolecule. More complicated molecules therefore seem to be needed, but then the number of combinations also become virtually infinitely large. It is in this connection interesting that new atomic negatively charged three-body structures recently were suggested [15].

Providing the Efimov conditions must be supplemented by production and detection of these states either by their decay or by increased scattering cross sections. The binding energies are exceedingly small and probably beyond the sensitivity and resolution of the experimental equipment. On the other hand the spatial extension seems to be a directly accessible observable. A grid with holes of variable sizes is a direct tool to determine the radius of the created three-body state [16]. The intensity of appropriate systems passing the grid then changes drastically when the Efimov states constitute a substantial part of the molecular bound states hitting the grid.

Detailed design of an experiment is beyond the scope of the present letter. However, a sketch of an experiment could be to let a beam of a Borromean system in its ground state pass into a region with an external electric field. By photon absorption the system must then be excited into the Efimov states which would be stopped at the grid. The Efimov effect might also manifest itself in a number of resonances in a three-body system which is slightly off the two-body threshold. Scattering experiments could then provide the decisive detection signals.

Conclusion. Weakly bound three-body systems interacting via short-range two-body potentials may exhibit spectacular properties exemplified by Borromean systems and the Efimov effect. The latter occurs when the scattering lengths are sufficiently large, or equivalently the virtual or bound two-body s-states are sufficiently close to zero. A number of excited states would then appear with very small binding energies and correspondingly large spatial extensions. The occurrence conditions for these Efimov states are rather sharply defined and even the lowest of these peculiar states is difficult to localize.

It would be a tremendous advantage if the appropriate two-body interactions could be adjusted to meet the necessary conditions. We propose an external electric field as the vehicle for fine tuning the effective two-body interaction to produce a zero energy s-state. The three-body computation with this field dependent two-body interaction then reveals the Efimov states at a particular field strength. As an illustration we used the atomic helium trimer systems. We estimated the field strength necessary to reach the occurrence conditions for the Efimov effect and found it on the limit of present day technology. However, it is conceivable that other systems are closer to the threshold and therefore are better suited candidates. Appropriate realistic systems should be searched for and investigated. We shall here be content with the demonstration that the idea of controlled creation of the occurrence conditions for the spectacular Efimov states is entirely feasible.

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[1] V. Efimov, Phys. Lett. **B33**, 563 (1970); Sov. J. Nucl. Phys. **12**, 589 (1971).

[2] D.V. Fedorov and A.S. Jensen, Phys. Rev. Lett. **71**, 4103 (1993).

[3] L.H. Thomas, Phys. Rev. **47**, 903 (1935).

[4] D.V. Fedorov, A.S. Jensen and K. Riisager, Phys. Rev. Lett. **73**, 2817 (1994).

[5] B.D. Esry, C.D. Lin and C.H. Greene, Phys. Rev. **A54**, 394 (1996).

[6] A.K. Motovilov, S.A. Sofianos and E.A. Kolganova, J. Phys. **B31**, 1279 (1998).

[7] E. Nielsen, D.V. Fedorov and A.S. Jensen, J. Phys. **B31**, 4085 (1998).

[8] J.M. Vogels, C.C. Tsai, R.S. Freeland, S.J.J.M.F Kokkelmans, B.J. Verhaar and D.J. Heinzen, Phys. Rev. **A56**, R1067 (1998).

[9] Ph. Courteille, R.S. Freeland, D.J. Heinzen, F.A. van Abeelen and B.J. Verhaar, Phys. Rev. Lett. **81**, 69 (1998).

[10] S. Inouye, M.R. Andrews, J. Stenger, H.-J. Miesner, D.M. Stamper-Kurn and W. Ketterle, Nature (London) **392**, 151 (1998).

[11] *CRC Handbook of Chemistry and Physics*, 72nd edition, CRC Press 1991, p. **10**-194.

[12] W. Ketterle and D.E. Pritchard, Applied Physics **B54**, 403 (1992).

[13] R.A. Aziz and M.J. Slaman, J. Chem. Phys. **94**, 8047 (1993).

[14] J.H. Posthumus, A.J. Giles, M.R. Thompson and K. Codling, J. Phys. **B29**, 5811 (1996).

[15] F. Robicheaux, Preprint, APS E-print database (1997), aps1997jul01001

[16] F. Lou, C.F. Giese, and W.R. Gentry, J. Chem. Phys **104**, 1151 (1996).